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The geometrical optics design of reflectors using complex coordinates

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Abstract. The mathematics of reflector design under the geometrical optics approximation is treated using complex coordinates. Conformal transformations occur naturally in the new formalism and their application to reflector design appears to have great potential. Some detailed illustrations are included.

1. Introduction

In a recent series of papers on reflector synthesis under the geometrical optics approximation two approaches have been used to establish design methods for arbitrary two-variable far-field specification. These methods depend on whether the partial differential equations governing the mapping τ between incident and reflected ray directions are elliptic or hyperbolic.

Westcott and Norris (1975) consider the elliptic case in which a boundary value problem is posed and sufficient conditions are derived for its solution. Numerical procedures are described in Norris and Westcott (1976).

The hyperbolic case is solved as an initial value problem in two papers by Brickell and Westcott (1976a, b). A numerical method and results are described in Westcott and Brickell (1976).

The present paper shows that the mathematics involved in the previous approaches can be simplified and unified by the use of complex coordinates. The new formalism makes obvious the fact that τ can be any analytic (in particular, conformal) mapping. Thus complex potential theory can be used in reflector design and the method appears to have exciting possibilities.

The paper is organized in the following way. In § 2 we explain our notation and introduce the idea of the distortion of a mapping τ of a unit sphere into itself. For example, a conformal mapping has uniform distortion. If the mapping τ is defined by a reflector then it must satisfy an integrability condition which is derived in § 3. Analytic mappings arise naturally because they automatically satisfy this condition. The other basic equations in the theory are also included in § 3.

It is important to express properties of the reflector in terms of the mapping τ. Thus in § 4 we find a formula for the positions of the caustic points on a reflected ray and, in § 5, we obtain a formula for the Gaussian curvature of the reflector. There are close relations between these and the distortion of τ.

In § 6 we examine a reciprocity that occurs in the general theory. If τ is a one–one mapping then its inverse also belongs to a reflector. This fact is useful in problems where it is natural to use the far-field coordinate as the independent one.

The last two sections are concerned with the application of complex potential theory to reflector design. Some particular models are worked out in detail.

2. Complex coordinates on the unit sphere

We begin by summarizing some standard notation associated with complex coordinates. A complex-valued function f of real variables x, y can be written $f = u + \mathrm{i}v$ where u, v are real-valued functions and $\mathrm{i} = \sqrt{-1}$. The partial derivatives are defined by $f_x = u_x + \mathrm{i}v_x$, $f_y = u_y + \mathrm{i}v_y$.

We can also regard f as a function of the complex variable $\eta = x + \mathrm{i}y$. The relations $2x = \eta + \bar{\eta}$, $2\mathrm{i}y = \eta - \bar{\eta}$ motivate the definitions

$$f_\eta = \tfrac{1}{2}(f_x - \mathrm{i}f_y) = \tfrac{1}{2}[u_x + v_y + \mathrm{i}(v_x - u_y)] \tag{1}$$

$$f_{\bar{\eta}} = \tfrac{1}{2}(f_x + \mathrm{i}f_y) = \tfrac{1}{2}[u_x - v_y + \mathrm{i}(v_x + u_y)]. \tag{2}$$

The equation $f_{\bar{\eta}} = 0$ is equivalent to the Cauchy–Riemann equations. Consequently a function f satisfying this condition is an analytic function of η. Similarly, if $f_\eta = 0$ then f is an analytic function of $\bar{\eta}$.

We denote the conjugate complex function $u - \mathrm{i}v$ by $\bar{f}$. Its derivatives satisfy the relations

$$\bar{f}_\eta = (\overline{f_{\bar{\eta}}}), \qquad \bar{f}_{\bar{\eta}} = (\overline{f_\eta}). \tag{3}$$

The commutativity relation $f_{\eta\bar{\eta}} = f_{\bar{\eta}\eta}$ follows from the equations (1) and (2), both of these derivatives being equal to one quarter of the Laplacian Δf.

The following lemma will be used in later work.

Lemma 1. Let f be a function with continuous derivatives of first order. There exists a real-valued function g such that $g_\eta = f$ if, and only if, $f_{\bar{\eta}}$ is real valued.

Proof. If g exists then $f_{\bar{\eta}} = g_{\eta\bar{\eta}} = \frac{1}{4}\Delta g$ and is therefore real valued. Conversely, if $f_{\bar{\eta}}$ is real valued then, from equation (1), $u_y = -v_x$. Consequently there exists a real-valued function ϕ such that $u = \phi_x$, $v = -\phi_y$. The function $g = 2\phi$ satisfies $g_\eta = f$.

We now explain a standard way of associating a complex coordinate with a point on a unit sphere. Let O denote the centre of the sphere and choose a rectangular set of axes OX, OY, OZ as in figure 1. Let (x, y, z) be the corresponding Cartesian coordinates. Under stereographic projection from the point N of coordinates (0, 0, 1) a point P on the unit sphere projects to a point P′ in the plane $z = 0$. The complex coordinate η of P is defined as $x + \mathrm{i}y$ where $(x, y, 0)$ are the Cartesian coordinates of P′.

The metric on the sphere assumes a simple form in terms of the coordinate η. To show this we first note that the position vector $\boldsymbol{p}$ of P satisfies

$$(1 + |\eta|^2)\boldsymbol{p} = (\eta + \bar{\eta}, \mathrm{i}(\bar{\eta} - \eta), |\eta|^2 - 1), \tag{4}$$

so that its derivative $\boldsymbol{p}_\eta$ is the complex-valued vector given by

$$(1 + |\eta|^2)^2\boldsymbol{p}_\eta = (1 - \bar{\eta}^2, -\mathrm{i}(1 + \bar{\eta}^2), 2\bar{\eta}).$$

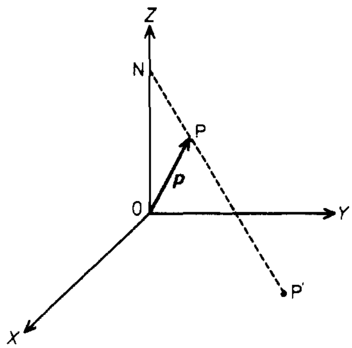

Figure 1. Diagram showing coordinate system.

It follows from equation (3) that $p_{\bar{\eta}} = \overline{(p_\eta)}$. These formulae enable us to calculate the scalar products

$$\begin{aligned} &\boldsymbol{p} \cdot \boldsymbol{p}_\eta = \boldsymbol{p} \cdot \boldsymbol{p}_{\bar{\eta}} = 0, \\ &\boldsymbol{p}_\eta \cdot \boldsymbol{p}_\eta = \boldsymbol{p}_{\bar{\eta}} \cdot \boldsymbol{p}_{\bar{\eta}} = 0, \qquad \boldsymbol{p}_\eta \cdot \boldsymbol{p}_{\bar{\eta}} = 2/(1+|\eta|^2)^2. \end{aligned} \tag{5}$$

Consider a curve $\eta = \eta(t)$ on the sphere where t is a real parameter. This curve has a tangent vector given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p}(\eta(t))) = \boldsymbol{p}_\eta \frac{\mathrm{d}\eta}{\mathrm{d}t} + \boldsymbol{p}_{\bar{\eta}} \frac{\mathrm{d}\bar{\eta}}{\mathrm{d}t}$$

and, using the equations (5), we obtain the promised simple formula for its length

$$\left|\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right| = 2\left|\frac{\mathrm{d}\eta}{\mathrm{d}t}\right|(1+|\eta|^2)^{-1}. \tag{6}$$

In the following sections we shall be concerned with a local mapping τ of the unit sphere into itself. Suppose that τ is given by $\zeta = \zeta(\eta)$ in terms of the complex coordinates η of P and ζ of $\mathrm{Q} = \tau(\mathrm{P})$. The curve $\eta = \eta(t)$ transforms into the curve $\zeta = \zeta(\eta(t))$ and according to (6), the length of the tangent vector to this curve is

$$2\left|\zeta_\eta \frac{\mathrm{d}\eta}{\mathrm{d}t} + \zeta_{\bar{\eta}} \frac{\mathrm{d}\bar{\eta}}{\mathrm{d}t}\right|(1+|\zeta|^2)^{-1}.$$

Using this formula, it can be shown that the circle of unit vectors tangent to the sphere at P transforms into an ellipse of tangent vectors at Q whose major and minor axes are of lengths

$$\frac{1+|\eta|^2}{1+|\zeta|^2}(\left||\zeta_\eta| \pm |\zeta_{\bar{\eta}}|\right|). \tag{7}$$

These numbers measure the distortion of τ. We say that τ has uniform distortion if they are equal. It follows that the mappings of uniform distortion satisfy either $\zeta_{\bar{\eta}} = 0$ or $\zeta_\eta = 0$. In the first case ζ is an analytic function of η in the sense of complex variable theory and we say that τ is analytic. In the second case we say that τ is anti-analytic.

It is a consequence of the formula (7) that τ alters areas by a factor

$$\left(\frac{1+|\eta|^2}{1+|\zeta|^2}\right)^2 |J(\tau)| \tag{8}$$

where $J(\tau)$, the Jacobian of τ is given by

$$J(\tau) = |\zeta_\eta|^2 - |\zeta_{\bar{\eta}}|^2.$$

3. Basic equations

In figure 2 a sphere of unit radius and centre O is drawn, where O is the point source of incident rays. The points P, Q are the end points of unit vectors $\boldsymbol{p}$, $\boldsymbol{q}$ drawn from O. The unit vector $\boldsymbol{p}$ is in the direction of the incident ray, $\boldsymbol{r} = r\boldsymbol{p}$ is the position vector of the point of reflection R, and $\boldsymbol{q}$ is in the direction of the reflected ray. Our aim is to relate the far-field power density pattern $G(\boldsymbol{q})$ and the source power pattern $I(\boldsymbol{p})$ to the geometry of the reflector.

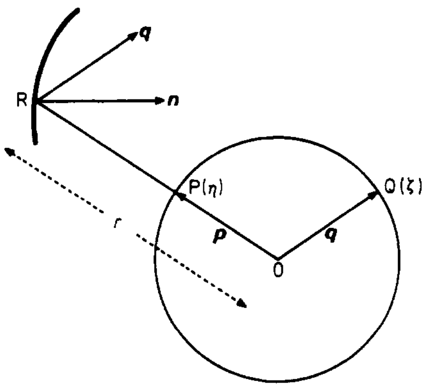

Figure 2. Diagram showing incident and reflected ray directions.

We obtain first of all the differential equations governing the mapping τ: P → Q. We shall use complex coordinates, the coordinates of P, Q being denoted by η, ζ respectively.

The relation between the power densities implies that τ has to alter areas by the factor $F = I/G$. We allow G to be infinite so that F can be zero. Equation (8) leads immediately to one of the differential equations

$$|\zeta_\eta|^2 - |\zeta_{\bar{\eta}}|^2 = \pm\left(\frac{1+|\zeta|^2}{1+|\eta|^2}\right)^2 F. \tag{9}$$

The differential equation (9) is not the only restriction on τ. We shall see that $\rho = \ln r$ satisfies a differential equation whose integrability condition gives a second condition on τ. The law of reflection implies that $\boldsymbol{r} - r\boldsymbol{q}$ is normal to the reflector surface and consequently

$$\boldsymbol{r}_\eta \cdot (\boldsymbol{r} - r\boldsymbol{q}) = 0.$$

This condition can be modified to give

$$\rho_\eta = (\boldsymbol{q} \cdot \boldsymbol{p}_\eta)/\Lambda$$

where $\Lambda(\eta, \zeta) = 1 - \boldsymbol{p} \cdot \boldsymbol{q}$. We put $\Psi(\eta, \zeta) = -\ln \Lambda$ so that we can write the above equation as

$$\rho_\eta = \Psi_\eta(\eta, \zeta(\eta)). \tag{10}$$

A calculation based on equation (4) shows that

$$\Lambda(\eta, \zeta) = 2|\zeta - \eta|^2/(1 + |\zeta|^2)(1 + |\eta|^2). \tag{11}$$

We differentiate to obtain Ψ_η and so express equation (10) as

$$\rho_\eta = \frac{1}{\zeta - \eta} + \frac{\bar{\eta}}{1 + |\eta|^2}.$$

It is convenient to write this result as

$$L_\eta = 1/(\zeta - \eta) \tag{12}$$

where $L(\eta) = \ln[r/(1 + |\eta|^2)]$. L is a real-valued function and consequently lemma 1 implies that

$$\frac{1}{(\zeta - \eta)^2} \zeta_{\bar{\eta}} \quad \text{is real valued.} \tag{13}$$

This is the second condition on the mapping τ.

Conversely, if τ is a local mapping of a sphere into itself satisfying the conditions (9) and (13), corresponding reflectors can be constructed. For, according to lemma 1, there exists a real-valued function $L(\eta)$ satisfying (12) and the reflector is given (to within a multiplicative constant) by

$$r = (1 + |\eta|^2) \exp L(\eta). \tag{14}$$

For a given non-zero F one can obtain two types of reflector depending on the choice of sign in (9). We shall refer to the choice of the $+(-)$ sign as the hyperbolic (elliptic) case respectively.

Finally in this section we show that the function L satisfies a partial differential equation. We can deduce from (12) that

$$L_{\eta\eta} - L_\eta^2 = -L_\eta^2 \zeta_\eta, \qquad L_{\eta\bar{\eta}} = -L_\eta^2 \zeta_{\bar{\eta}}.$$

Consequently we obtain from equation (9)

$$|L_{\eta\eta} - L_\eta^2|^2 - |L_{\eta\bar{\eta}}|^2 = \pm |L_\eta|^4 \left(\frac{1 + |\zeta|^2}{1 + |\eta|^2}\right)^2 F. \tag{15}$$

When the variable ζ is replaced by $\eta + (1/L_\eta)$ this equation becomes a partial differential equation of second order for L in terms of η. It is a Monge–Ampère equation of hyperbolic or elliptic type depending on the choice of sign.

It is obvious that any analytic mapping of the unit sphere into itself satisfies condition (13) and therefore belongs to a reflector. A corresponding function F can be obtained from equation (9) (with the + sign). This remark may be the basis of a powerful method for designing reflectors giving prescribed intensity peaks in the far field. We shall discuss this further in §§ 7 and 8.

We will say that a reflector has uniform distortion if the corresponding mapping τ has uniform distortion. A property of reflectors designed using analytic mappings is that they all have uniform distortion. It is natural to ask if there are other reflectors with this property. According to the work in § 1 τ has to be anti-analytic, that is $\zeta_\eta = 0$. In this case condition (13) is not automatically satisfied and imposes severe restrictions on τ. It can be shown that τ is necessarily of the form

$$\zeta = (a\bar{\eta} + b)/(c\bar{\eta} + d)$$

where a, b, c, d are constants satisfying certain further restrictions.

4. Caustics

The caustic surface of a reflector is the locus of the points where the intensity is infinite. There are two such caustic points on each reflected ray, although one or both of them may be virtual, that is situated behind the reflector. We shall obtain a formula for the positions of these points in terms of the mapping τ.

In our calculations we shall use the function Ψ introduced previously. We have already seen that

$$\Psi_\eta = \frac{1}{\zeta - \eta} + \frac{\bar{\eta}}{1 + |\eta|^2} = (\boldsymbol{q} \cdot \boldsymbol{p}_\eta)/\Lambda. \tag{16}$$

Similarly

$$\Psi_\zeta = \frac{1}{\eta - \zeta} + \frac{\bar{\zeta}}{1 + |\zeta|^2} = (\boldsymbol{p} \cdot \boldsymbol{q}_\zeta)/\Lambda. \tag{17}$$

As Ψ is real we can obtain the derivatives $\Psi_{\bar{\eta}}$, $\Psi_{\bar{\zeta}}$ by taking complex conjugates (see equation (3)). We also need the following formulae which can be obtained from equation (16):

$$\Psi_{\eta\zeta} = -\frac{1}{(\zeta - \eta)^2} = \Psi_\eta \Psi_\zeta + (\boldsymbol{p}_\eta \cdot \boldsymbol{q}_\zeta)/\Lambda, \tag{18}$$

$$\Psi_{\eta\bar{\zeta}} = 0 = \Psi_\eta \Psi_{\bar{\zeta}} + (\boldsymbol{p}_\eta \cdot \boldsymbol{q}_{\bar{\zeta}})/\Lambda. \tag{19}$$

To obtain the caustic points we introduce the function

$$\boldsymbol{w}(\eta, \lambda) = \boldsymbol{r}(\eta) + (\lambda - r(\eta))\boldsymbol{q}(\zeta(\eta)).$$

As λ varies the point whose position vector is $\boldsymbol{w}(\eta, \lambda)$ describes the reflected ray corresponding to the incident ray of coordinate η. In general, given a point of position vector $\boldsymbol{s}$ there are unique values of η, λ for which $\boldsymbol{w}(\eta, \lambda) = \boldsymbol{s}$. The caustic points are those for which this uniqueness breaks down. Consequently the caustic points are given by those values of η, λ for which the vectors $\boldsymbol{w}_\eta$, $\boldsymbol{w}_{\bar{\eta}}$ and $\boldsymbol{w}_\lambda = \boldsymbol{q}$ are linearly dependent.

We have to calculate $\boldsymbol{w}_\eta$. Using equation (10) we find that

$$\boldsymbol{w}_\eta = r(\Psi_\eta(\boldsymbol{p} - \boldsymbol{q}) + \boldsymbol{p}_\eta) + (\lambda - r)(\zeta_\eta \boldsymbol{q}_\zeta + \bar{\zeta}_\eta \boldsymbol{q}_{\bar{\zeta}}). \tag{20}$$

The relations (5) (but with $\boldsymbol{p}$, η replaced by $\boldsymbol{q}$, ζ) together with equation (16) show that

$$\boldsymbol{w}_\eta \cdot \boldsymbol{q} = r(-\Lambda\Psi_\eta + \boldsymbol{p}_\eta \cdot \boldsymbol{q}) = 0, \qquad \boldsymbol{w}_{\bar{\eta}} \cdot \boldsymbol{q} = 0.$$

It follows that the condition for the caustic points reduces to the linear dependence of the vectors $\boldsymbol{w}_\eta$, $\boldsymbol{w}_{\bar\eta}$. It also follows that we can write

$$\boldsymbol{w}_\eta = a\boldsymbol{q}_\zeta + b\boldsymbol{q}_{\bar\zeta}, \qquad \boldsymbol{w}_{\bar\eta} = \bar{a}\boldsymbol{q}_{\bar\zeta} + \bar{b}\boldsymbol{q}_\zeta. \tag{21}$$

Thus the caustic points are determined by the condition

$$|a| = |b| \tag{22}$$

and our task is reduced to calculating expressions for a and b.

By using the appropriate scalar products on equation (21) we find that

$$2a = (\boldsymbol{w}_\eta \cdot \boldsymbol{q}_{\bar\zeta})(1+|\zeta|^2)^2, \qquad 2b = (\boldsymbol{w}_\eta \cdot \boldsymbol{q}_\zeta)(1+|\zeta|^2)^2.$$

Then a calculation starting from equation (20) and making use of the formulae (5), (17), (18), (19) and (11) shows that

$$a = (\lambda - r)\zeta_\eta, \qquad b = (\lambda - r)\bar\zeta_\eta - r\frac{|\zeta-\eta|^2}{(\zeta-\eta)^2}\left(\frac{1+|\zeta|^2}{1+|\eta|^2}\right).$$

We shall modify the formula for b in order to introduce $\zeta_{\bar\eta}$ instead of $\bar\zeta_\eta$. This is possible because the integrability condition (13) implies that

$$\bar\zeta_\eta/(\bar\zeta-\bar\eta)^2 = \zeta_{\bar\eta}/(\zeta-\eta)^2$$

and consequently

$$\bar\zeta_\eta = |\zeta-\eta|^4\zeta_{\bar\eta}/(\zeta-\eta)^4.$$

At this stage it is convenient to define the expressions

$$\Sigma = \left(\frac{1+|\eta|^2}{1+|\zeta|^2}\right)\frac{|\zeta-\eta|^2}{(\zeta-\eta)^2}\zeta_{\bar\eta}, \qquad \Sigma' = \left(\frac{1+|\eta|^2}{1+|\zeta|^2}\right)\frac{|\zeta-\eta|^2}{(\zeta-\eta)^2}\zeta_\eta. \tag{23}$$

The expression Σ (which is real valued) can be introduced into the formula for b to give

$$b = \left(\frac{1+|\zeta|^2}{1+|\eta|^2}\right)\frac{|\zeta-\eta|^2}{(\zeta-\eta)^2}[(\lambda-r)\Sigma - r].$$

We can now obtain our formula for the positions of the caustic points. Suppose that a caustic point lies on a reflected ray at a distance κr from the point of reflection. Remembering that Σ is real valued we find from equation (22) that

$$\kappa = 1/(\Sigma \pm |\Sigma'|).$$

We denote these values by κ_1, κ_2. It follows from equation (9) that

$$\frac{1}{\kappa_1\kappa_2} = \mp F, \qquad \frac{1}{\kappa_1} + \frac{1}{\kappa_2} = 2\Sigma. \tag{24}$$

The caustic points are real or virtual according as the values of κ are positive or negative. Thus in the hyperbolic case there is just one real caustic point on each reflected ray. In the elliptic case there are two if $\Sigma > 0$ and none if $\Sigma < 0$.

We also point out that the absolute values $|\kappa_1|$, $|\kappa_2|$ are the reciprocals of the extreme values of the distortion of τ (see equation (7)).

5. Reflector curvature

There is a simple formula for the Gaussian curvature of the reflector surface in terms of the mapping τ. We shall outline the calculations involved in obtaining this formula.

The matrix γ of the first fundamental form of the reflector surface is defined by

$$\gamma = \begin{bmatrix} \boldsymbol{r}_\eta \cdot \boldsymbol{r}_\eta & \boldsymbol{r}_\eta \cdot \boldsymbol{r}_{\bar\eta} \\ \boldsymbol{r}_\eta \cdot \boldsymbol{r}_{\bar\eta} & \boldsymbol{r}_{\bar\eta} \cdot \boldsymbol{r}_{\bar\eta} \end{bmatrix}.$$

The matrix $\mathbf{H}$ of the second fundamental form is defined by

$$\mathbf{H} = -\begin{bmatrix} \boldsymbol{r}_\eta \cdot \boldsymbol{n}_\eta & \boldsymbol{r}_\eta \cdot \boldsymbol{n}_{\bar\eta} \\ \boldsymbol{r}_{\bar\eta} \cdot \boldsymbol{n}_\eta & \boldsymbol{r}_{\bar\eta} \cdot \boldsymbol{n}_{\bar\eta} \end{bmatrix}$$

where $\boldsymbol{n}$, the unit normal vector to the reflector surface is given by

$$\boldsymbol{n} = (\boldsymbol{q} - \boldsymbol{p})/(2\Lambda)^{1/2}.$$

The Gaussian curvature of the reflector is equal to $\det \mathbf{H}/\det \gamma$. We remark that positive Gaussian curvature at a point R implies that, in a neighbourhood of R, the surface lies entirely on one side of the tangent plane at R. Negative Gaussian curvature implies that the surface lies on both sides of the tangent plane.

Calculations similar to those in § 3 lead to the formulae

$$\gamma = r^2 \begin{bmatrix} \Psi_\eta^2 & \Psi_\eta \Psi_{\bar\eta} + \dfrac{2}{(1+|\eta|^2)^2} \\ \Psi_\eta \Psi_{\bar\eta} + \dfrac{2}{(1+|\eta|^2)^2} & \Psi_{\bar\eta}^2 \end{bmatrix}$$

$$\mathbf{H} = \frac{2r}{(2\Lambda)^{1/2}(1+|\eta|^2)^2} \begin{bmatrix} \Sigma' & 1+\Sigma \\ 1+\Sigma & \bar\Sigma' \end{bmatrix}$$

where Σ, Σ' are defined by the equations (23). It is then easy to show that the Gaussian curvature of the reflector surface is equal to

$$(1 \mp F + 2\Sigma)/4r^2. \tag{25}$$

We point out that we can use the equations (24) to write this expression as

$$\frac{1}{4r^2}\left(1+\frac{1}{\kappa_1}\right)\left(1+\frac{1}{\kappa_2}\right).$$

6. The reciprocal reflector

In this section we shall suppose that the function $F = I/G$ is non-zero. Equation (9) shows that the Jacobian $J(\tau)$ is non-zero and therefore, at least locally, τ admits an inverse mapping χ. Consequently we may use ζ as the independent variable. We find that

$$J(\chi) = 1/J(\tau), \qquad \zeta_{\bar\eta} = -\eta_{\bar\zeta} J(\tau)$$

so that the equation (9) and the condition (13) can be expressed as

$$|\eta_\zeta|^2-|\eta_{\bar\zeta}|^2=\pm\left(\frac{1+|\eta|^2}{1+|\zeta|^2}\right)^2 D$$

and

$$\frac{1}{(\eta-\zeta)^2}\eta_{\bar\zeta} \text{ is real valued}$$

where $D=G/I$.

It is clear from these relations that the mapping χ corresponds to a *reciprocal* reflector for which $\boldsymbol{q}$ is in the direction of the *incident* ray and $\boldsymbol{p}$ is in the direction of the reflected ray. The function $G(\zeta)$ becomes the incident field intensity and $I(\eta)$ becomes the far field intensity.

Let $\tilde r(\zeta)$ be the length of the radius vector of this reciprocal reflector. The real-valued function $M(\zeta)=\ln[\tilde r/(1+|\zeta|^2)]$ will satisfy $M_\zeta=1/(\eta-\zeta)$ which corresponds to equation (12). The Monge–Ampère equation corresponding to equation (15) is

$$|M_{\zeta\zeta}-M_\zeta^2|^2-|M_{\zeta\bar\zeta}|^2=\pm|M_\zeta|^4\left(\frac{1+|\eta|^2}{1+|\zeta|^2}\right)^2 D \tag{26}$$

where the variable η has to be replaced by $\zeta+(1/M_\zeta)$.

In some problems of reflector design it is convenient to work with the far-field variable ζ as the independent variable. Then the equations in this section replace those in § 2. Although they do not use complex coordinates this is essentially the method used by Westcott and Norris (1975). They introduce a function p (defined by equation (13) in their paper) and show that p satisfies a Monge–Ampère equation. It can be shown that p is just the function $(\tilde r)^{-1}$. The Monge–Ampère equation satisfied by p can be deduced from our equation (26).

7. Applications of conformal mapping

In this section we exploit the fact that every analytic function $\zeta(\eta)$ generates a reflector, for which (9) determines the power density ratio

$$D=GI^{-1}=F^{-1}.$$

The mapping τ is conformal when $\zeta_\eta\neq 0$ and this suggests an approach to reflector design using standard properties of conformal maps. The question arises: given $D(\eta,\zeta)$, can we determine $\zeta(\eta)$ satisfying (9) with $\zeta_{\bar\eta}=0$, $\zeta_\eta\neq 0$? Two theorems in complex variable theory indicate that, *as it stands*, such an approach is too restrictive to have wide application.

Theorem 1: maximum–minimum modulus principle. Let the function $f(\eta)$ be analytic on a domain which comprises a simple closed curve C and its interior. Let $f(\eta)\neq 0$ inside C. The maximum and minimum values of $|f(\eta)|$ in the domain occur on C.

Theorem 2: Koebe's distortion theorem. If $\zeta(\eta)$ is analytic and $\zeta_\eta\neq 0$ on the disc $|\eta|<1$, then all boundary points of the image domain under the mapping $\eta\to\zeta(\eta)$ are at a distance of at least $\frac{1}{4}|\zeta_\eta(0)|$ from $\zeta(0)$ (see Bieberbach 1953).

Theorem 1 (with $f = \zeta_\eta$) effectively precludes the occurrence of peaks and troughs in D at interior points of the far-field power density pattern. And Koebe's theorem sets a lower limit on possible values of D on the central ray incident on a reflector which subtends a solid circular cone at the source, if the boundary of the far-field cone is given.

To overcome these constraints we can generalize in two ways. First, we may allow $\zeta(\eta)$ to possess poles (ζ is *meromorphic* but not necessarily analytic) or branching singularities, with the proviso that ζ is to be rendered single valued where necessary by choice of branch. Physically, the latter statement means that each incident ray is reflected in one, and only one, direction. Secondly, we observe that for practical purposes the requirement (in fact not made in §§1–4) that τ have a unique inverse is unrealistic. (The simplest case of a many–one mapping arises in the production of a parallel beam by a parabolic reflector.) Thus, ζ_η may possess zeros.

Finally we remark that in the case of uniform distortion our theory permits the identification of ζ as a complex potential for a two-dimensional vector field. Used in conjunction with the existing extensive literature of classical field theory it may provide a powerful tool for use in two-variable far-field reflector design, and we suggest that this method of attack lends itself to much further development.

Some models are considered in the next section. In each instance the cone of the reflector is assumed circular and the central ray is reflected through a right angle. We adopt a *standard orientation* for the reflector relative to the Cartesian axes $OXYZ$, in which the central ray is incident along $-OZ$ and is reflected in the direction OY. Primed variables are used consistently for the standard orientation. For example, η, ζ are related to the spherical polar coordinates (α, β), (θ, ϕ) of the points P, Q on the unit sphere by the formulae

$$\eta = \cot(\tfrac{1}{2}\alpha)\, e^{i\beta}, \qquad \zeta = \cot(\tfrac{1}{2}\theta)\, e^{i\phi} \tag{27}$$

and the last condition is that $\alpha' = \pi \to \theta' = \frac{1}{2}\pi$, $\phi' = \frac{1}{2}\pi$, or equivalently $\eta' = 0 \to \zeta' = i$. The boundary of the reflector cone has an equation of the form $\alpha' = \alpha_r'$ or $|\eta'| = c = \cot(\frac{1}{2}\alpha_r')$ (figure 3).

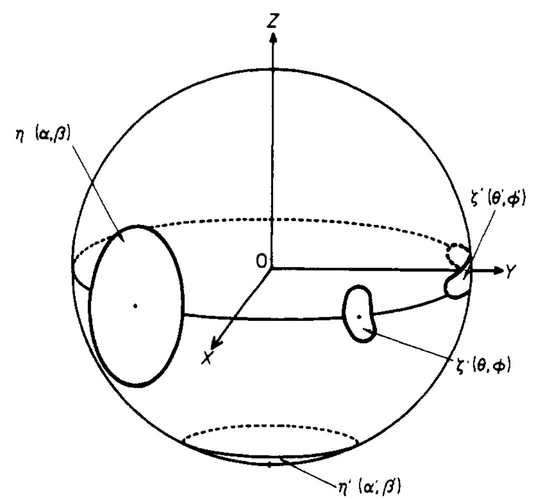

Figure 3. Domains of coordinates.

A re-orientation of the reflector and source by rotation about an axis through O is achieved by a Moebius transformation on the sphere of the form

$$\eta' \to \eta = \frac{a\eta' + b}{-\bar{b}\eta' + \bar{a}}, \qquad \zeta' \to \zeta = \frac{a\zeta' + b}{-\bar{b}\zeta' + \bar{a}} \tag{28}$$

where the constants a, b satisfy the normalization $|a|^2 + |b|^2 = 1$. Because the Moebius transformation is itself conformal, the mapping between domains on the sphere remains conformal under re-orientation. The source and far-field patterns undergo the same rigid motion on the sphere, and hence we have $D'(\eta', \zeta') = D(\eta, \zeta)$.

For computational purposes we have found useful the re-orientation

$$\eta = \frac{1-\mathrm{i}}{\surd 2}\,\frac{1+\eta'}{1-\eta'}, \qquad \zeta = \frac{1-\mathrm{i}}{\surd 2}\,\frac{1+\zeta'}{1-\zeta'}. \tag{29}$$

Equation (29) represents a rotation $-\frac{1}{2}\pi$ of the system about OY, followed by a rotation $-\frac{1}{4}\pi$ about OZ, and is derived from (28) by the conditions $0 \to (1-\mathrm{i})/\surd 2$, $\mathrm{i} \to (1+\mathrm{i})/\surd 2$.

Domains of the coordinates (relating to the example in § 8.2) are shown in figure 3. A function $\zeta'(\eta')$ defines a model in standard orientation, and $\theta(\alpha, \beta)$, $\phi(\alpha, \beta)$ may be obtained from (27), (29).

The equation of the reflector surface is obtained using (14).

8. Some examples

8.1. The parallel beam

In standard orientation the reflected rays are represented on the unit sphere by the single point $\zeta' = \mathrm{i}$ which is also the equation of the mapping τ, and the cone of incident rays is represented by $|\eta'| \leqslant c$. Multiple reflections are avoided if the domains of η', ζ' do not overlap, i.e. if $0 < c < 1$.

By (9), $D'(\eta', \mathrm{i})$ is infinite. In the geometrical optics approximation this situation is permissible because D is a ratio of energy area *densities* rather than of net energies in the far-field and incident domains, and the conservation of total energy is assured for any mapping by virtue of an integrated form of the area-relating property (8).

The equation of the reflector surface is

$$r' = l(1 + |\eta'|^2)\exp\left(2\,\mathrm{Re}\int_0^{\eta'} \frac{\mathrm{d}\eta''}{\mathrm{i} - \eta''}\right) \tag{30}$$

according to (12), (14), where l is a positive real constant. The integrand is analytic because the domain of η' excludes the point i. On performing the integration, (30) becomes

$$r' = l(1 + |\eta'|^2)[1 + \mathrm{i}(\eta' - \bar{\eta}') + |\eta'|^2]^{-1}$$

which can be written using (27)

$$l/r' = 1 - \sin\alpha' \sin\beta',$$

the equation of a paraboloid of revolution with semi-latus-rectum l, axis OY and vertex $x = 0$, $y = -\frac{1}{2}l$, $z = 0$.

The Gaussian curvature of the surface is given by (25) with $F=0$ and (because of (23)) $\Sigma=0$,

$$K_G = 1/4r'^2.$$

8.2. The complex potential

$$\zeta' = \mathrm{i} + K^{-1} \ln[(a+\eta')/(a-\eta')]. \tag{31}$$

In two-dimensional field theory, if K, a are constants (K real; $K, a \neq 0$) and the principal value of the logarithm is taken, (31) represents the vector field due to a line source/sink pair situated at the points $\eta' = \pm a$, where $\zeta'_{\eta'}$ has poles and the field strength $|\zeta'_{\eta'}|$ is infinite. Regarded as a mapping τ for the reflector in standard configuration we have $D'(\pm a, \zeta'(\pm a)) = 0$ if the points $\pm a$ lie within the domain of η'.

Write $\eta' = \lambda a\, \mathrm{e}^{\mathrm{i}\beta'}$, where we take λ, a real and positive, so that $0 \leqslant \lambda \leqslant c/a$, $0 \leqslant \beta' < 2\pi$. Equation (31) becomes, with $\zeta' = x' + \mathrm{i}y'$,

$$\exp\{K[x' + \mathrm{i}(y'-1)]\} = \frac{1+\lambda\, \mathrm{e}^{\mathrm{i}\beta'}}{1-\lambda\, \mathrm{e}^{\mathrm{i}\beta'}}. \tag{32}$$

If $c/a \geqslant 1$, real values of η' map onto a region which contains the entire line $y'-1=0$ and therefore the maximum angular diameter of the cone of reflected rays is at least 180°. We shall exclude this case, setting $c < a$. In particular, for definiteness we have made numerical calculations for the model where $c = 0{\cdot}268$ (corresponding to a reflector cone semi-angle of 30°) and $a = 0{\cdot}27$.

By (32), the mapping is symmetric about the lines $x'=0$, $y'=1$ in the complex plane, the ζ' domain being roughly elliptical with 'minor axis' along $y'=1$. The most interesting feature of the model is the existence of two peaks in the value of D' at the extremities of the minor axis, at the points

$$x'=0, \qquad y' = y_0' = 1 \pm K^{-1} \tan^{-1}\left(\frac{2ac}{a^2-c^2}\right) = 1 \pm 1{\cdot}563\, K^{-1}. \tag{33}$$

The constant K now governs the scale of the system. A diagram for the case $K=22$, chosen to produce an angular separation of 8° between the peaks on the far-field sphere is shown in figure 4, and is typical. Symmetry in the complex plane is not acccurately preserved under stereographic projection onto the unit sphere.

The peak values of D may be expressed in terms of the value D_0 on the central ray ($\theta' = \phi' = \frac{1}{2}\pi$). They are given by

$$\frac{D_{\text{peak}}}{D_0} = \left(\frac{(1+y_0'^2)(1+(c/a)^2)}{2(1+c^2)}\right)^2 = 3{\cdot}954 \text{ or } 2{\cdot}977,$$

the larger value relating to the positive sign in (33).

Reflector cross sections are shown in figures 5 and 6.

8.3. A model with n peaks at interior points

The mapping τ:

$$\zeta'_{\eta'} = K(\eta'-a_1)(\eta'-a_2)\ldots(\eta'-a_n), \tag{34}$$

where the a are distinct constants with $|a_p| \leqslant c$ ($p = 1, 2, \ldots, n$) and K is an arbitrary non-zero constant, defines a system in which D is infinite at the n prescribed points

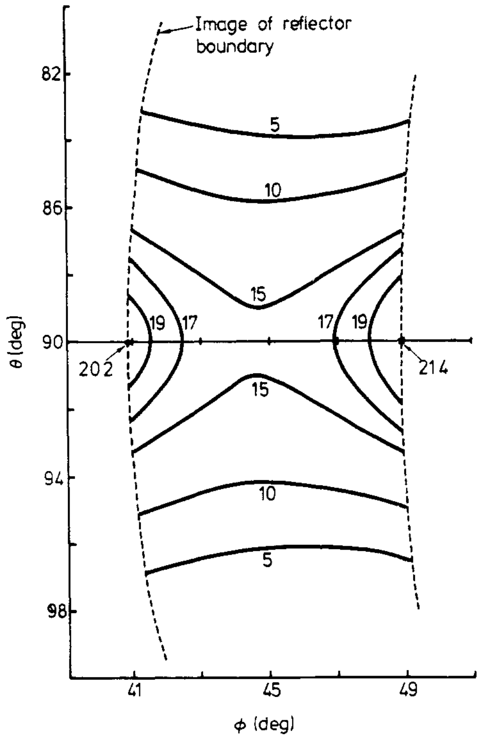

Figure 4. Constant-D contours (labelled in decibels) against reflected ray direction for the example of § 8.2.

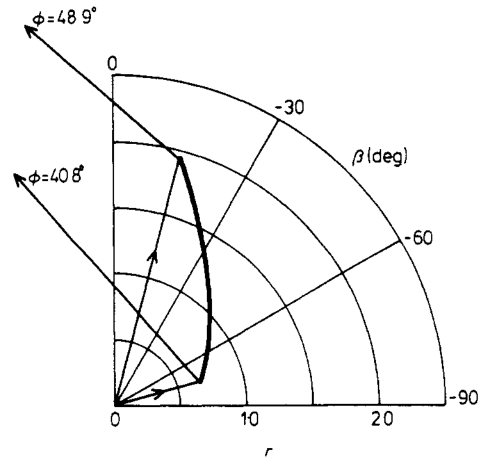

Figure 5. Central reflector cross section ($\alpha = \pi/2$) for the example of §8.2 showing edge rays.

$\eta' = a_p$ in the incident power density pattern. The far-field images of these points are

$$\zeta'_{(p)} = \mathrm{i} + K \int_0^{a_p} \prod_{q=1}^{n} (\eta'' - a_q)\, \mathrm{d}\eta''. \tag{35}$$

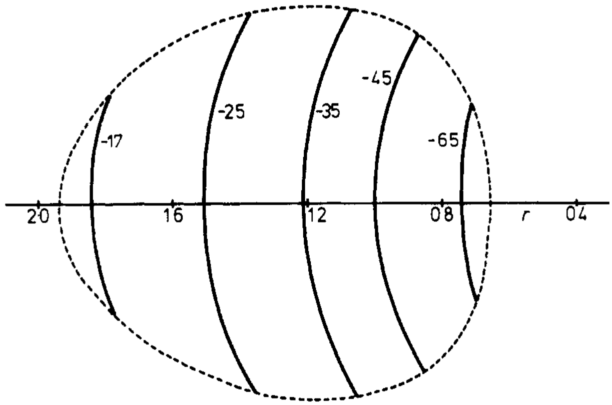

Figure 6. Constant-β reflector cross sections for the example of § 8.2. The full curves are labelled for values of β(deg).

We consider the case $n = 3$, with

$$a_p = a \exp[2(p-1)\pi i/3] \qquad (p = 1, 2, 3)$$

where a is a non-zero constant, so that (34) becomes

$$\zeta'_{\eta'} = K(\eta'^3 - a^3)$$

and the mapping τ is $\eta' \to \zeta'$:

$$\zeta' = i + \tfrac{1}{4}K\eta'(\eta'^3 - 4a^3).$$

The image points of $\eta' = a_p$ lie symmetrically on a circle in the complex plane, being given by $\zeta'_{(p)} - i = -\frac{3}{4}Ka^3 a_p$.

The constants K and a determine the orientation and scale of the far-field beam pattern as well as (for example) D_0 on the central ray $\zeta' = \zeta'(0) = i$, which also governs taper. Choosing one peak, $\zeta'_{(1)}$, to correspond to the point $\theta' = \frac{1}{2}\pi + \delta$, $\phi' = \frac{1}{2}\pi$, we arrange for this peak to be at angular distance $\delta (\delta > 0)$ from the reflected central ray. Thus,

$$\zeta'_{(1)} = i \cot(\tfrac{1}{4}\pi + \tfrac{1}{2}\delta) = (1 - \tfrac{3}{4}|Ka^4|)i$$

and

$$D_0 = \frac{4}{|\zeta'_{\eta'}(0)|^2} = \frac{4}{|K^2 a^6|},$$

which determine $|K|$, $|a|$ in terms of D_0, δ, and require further that Ka^4 be positive imaginary. The only remaining freedom lies in the orientation of a_1 on the circle $|\eta'| = |a|$, and is of little interest.

Figure 7 shows the D contours in the beam pattern, following re-orientation (29), in terms of θ, ϕ for the case $K = 14{\cdot}2i$, $a = -0{\cdot}285i$, for which δ is approximately 4°. The reflector cone semi-angle (which must in any case exceed $2 \tan^{-1}|a|$) is taken to be 35°. Within the three loops in the image of the reflector cone boundary the inverse mapping has two branches; elsewhere it is single valued. Use of (25) shows that the Gaussian curvature is finite at each point of the reflector surface. Additionally, we have calculated the principal curvatures and find that these, too, are finite, The calculation is

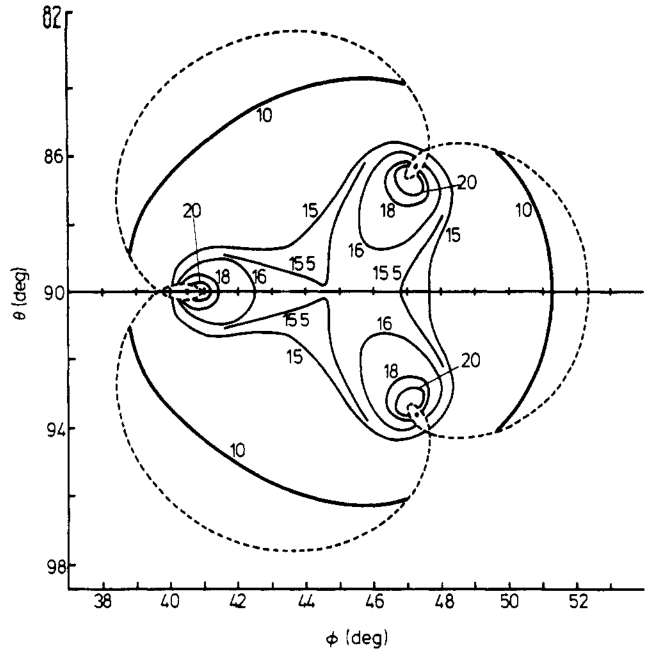

Figure 7. Constant-D contours (labelled in decibels) against reflected ray direction for the example of § 8.3.

lengthy and is omitted. The central reflector cross section ($\alpha = 90°$) is drawn in figure 8, and the departure of the β-constant cross sections from arcs of circles is shown in figure 9, by plotting $(r - r_{(\alpha=\frac{1}{2}\pi)})/r_{(\alpha=\frac{1}{2}\pi)}$ against α.

An outstanding problem, now under active consideration, is the following generalization of this model for arbitrary n. Can a mapping be found which produces infinite peaks in D, with arbitrary taper, at selected points in the far-field power density pattern?

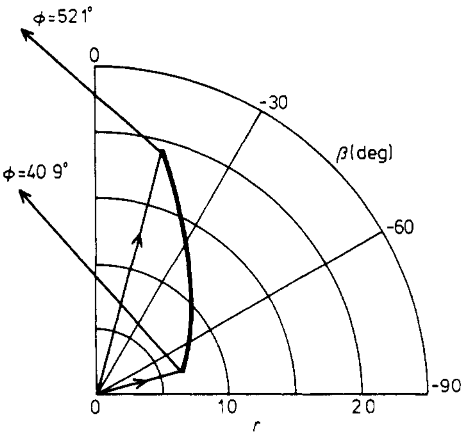

Figure 8. Central reflector cross section ($\alpha = \pi/2$) for the example of § 8.3 showing edge rays.

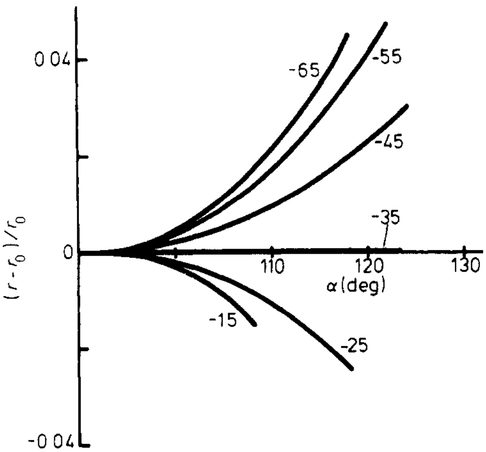

Figure 9. Departure of constant-β reflector cross sections from circular arcs. The full curves are labelled for values of β(deg); $r_0 = r_{(\alpha=\frac{1}{2}\pi)}$.

8.4. An anti-conformal example: the quadric of revolution

The equation

$$l/r' = 1 - e \sin \alpha' \sin \beta' \tag{36}$$

represents a quadric of revolution with its major axis (axis of symmetry) aligned along OY, with one focus at O and eccentricity e ($e > 0$). Except in the case $e = 1$ (the example of §8.1) a reflecting surface of this class does not correspond to a conformal mapping τ. The result follows by consideration of (12) and (14) and because the real-valued function $L(\eta') = \ln[r'/(1+|\eta'|^2)]$ is harmonic in the conformal case; a simple calculation using (27) shows that this obtains only when $e = 1$. Indeed, when the primed coordinates α', β' are expressed in terms of η' using (12) we find that

$$L_{\eta'} = (\zeta' - \eta')^{-1} = -(\bar{\eta}' + \mathrm{i}e)[1 + |\eta'|^2 + \mathrm{i}e(\eta' - \bar{\eta}')]^{-1}$$

or

$$\zeta' = \frac{\mathrm{i}e\bar{\eta}' - 1}{\bar{\eta}' + \mathrm{i}e}$$

which shows explicitly that the mapping is always *anti*-conformal. (The orientation of the quadric has been chosen for simplicity. For $e \neq 1$ the orientation is not standard, however, because $\zeta'(0) \neq \mathrm{i}$.) The mapping is always one of uniform distortion (see § 3).

The spherical reflector about O is a special case of (36), with $e = 0$, and the last equation reduces to $\zeta' = -1/\bar{\eta}'$, confirming that incident and reflected rays occupy antipodal points on the unit sphere.

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